Stationary dark localized modes: Discrete nonlinear Schrödinger equations

V. V. Konotop[1] and S. Takeno[2]
[1]Department of Physics and Center of Mathematical Sciences, University of Madeira, Praça do Município, Funchal P-9000, Portugal
[2]Faculty of Information Science, Osaka Institute of Technology, 1-79-1 Hirakata, Osaka 573-01, Japan
(Received 21 December 1998)

Various kinds of stationary dark localized modes in discrete nonlinear Schrödinger equations are considered. A criterion for the existence of such excitations is introduced and an estimation of a localization region is provided. The results are illustrated in examples of the deformable discrete nonlinear Schrödinger equation, of the model of Frenkel excitons in a chain of two-level atoms, and of the model of a one-dimensional Heisenberg ferromagnetic in the stationary phase approximation. The three models display essentially different properties. It is shown that at an arbitrary amplitude of the background it is impossible to reach strong localization of dark modes. In the meantime, in the model of Frenkel excitons, exact dark compacton solutions are found. [S1063-651X(99)03307-3]

PACS number(s): 45.05.+x, 63.20.Pw, 63.20.Ry, 05.45.Yv

I. INTRODUCTION

During recent years a great deal of attention was paid to dynamical properties of nonlinear lattices in which there exists a diversity of excitations characterized by the energy localized in space. Considering relative displacements of the neighbor sites as generalized coordinates, one can speak about the spatial localization of the displacements themselves. Such objects have been given various names. Being localized on a relatively large number of atoms, they are called solitons (or envelope solitons) [1]. In the integrable model one uses the name soliton in the restricted mathematical sense [2,3]. When excitations are localized on a very few atoms, they are called intrinsic localized modes [4–6]. Periodic in time and localized in space, the excitations are also called breathers. The existence and stability of breathers in a rigorous mathematical sense, in the so-called unticontinuum limit (i.e., in the limit when coupling between two nearest neighbors is small enough), has been proven in [7]. The discussion of the relation between the intrinsic localized modes and envelope solitons can be found in [8].

Considering the real displacements, rather than relative ones, one meets other names in the literature. So, *kink* is used for a real solution which amplitude goes to some constants at the infinity. Similar solutions, but associated with integrable models and generalized to complex fields, are called *dark solitons* [9]. The energy of dark solitons and kinks is still localized in space. Although formally by passing to relative displacements one can reduce a dynamical problem with homogeneous nonzero boundary conditions to one with the displacement field tending to zero at infinity, properties of dark solitons are very different form properties of bright ones (hereafter this last name will be used for localized solutions subject to zero boundary conditions). The mentioned difference is well studied for integrable models.

One of the physical reasons for the essential distinction between dark and bright solitons is that the frequency of the carrier wave (cw) in the former case is not arbitrary [or more precisely, is not determined by the cw vector, which is arbitrarily chosen from the first Brillouin zone (BZ)]. Instead, the frequency is determined by the amplitude of the background (hereafter the background refers to a solution of a nonlinear dynamical problem having constant amplitude which is defined by the boundary conditions). So, for example, it is well known that the dark soltion [9] of the Ablowitz-Ladik (AL) model [3] is determined by one essential parameter giving both the width and the amplitude of a soliton, while the bright soliton is determined by two essential parameters associated with the amplitude and the velocity of the wave [3].

The aim of the present paper is to elaborate a theory of *dark modes* (DLM) localized on a few atoms. We will concentrate on stationary excitations which do not display forward motion. It is to be mentioned here that examples of dark modes have been observed experimentally [10]. Various approximations for obtaining bright localized modes are well elaborated. One of them is a self-consistent theory based on the Green-function approach [5]. Mathematically, the frequency detuning the allowed band outwards, being large enough, allows one to consider the Green function to be localized on a few sites and to reduce the respective lattice sums to a very few terms. The straightforward application of the Green-function method to the dark-mode problem fails since the cw frequency is detuned to the allowed band and thus the Green function turns out to be delocalized. On the other hand, introducing a relative displacement of two nearest neighbors results in nonlocal terms in cases when on-site nonlinearities are included in the consideration. Another approach using special scaling and valid only for special points in the BZ [they are $\pm\pi/(3a)$ and $\pm 2\pi/(3a)$, a being a lattice constant] has been elaborated in [6]. That approach allows one to obtain movable intrinsic localized modes by reducing the evolution equation to the AL model [3]. Although the mentioned approximation yields a wave packet with stronger localization than the envelope soliton, it is still not applicable for strongly localized modes.

Speaking about physical applications of the DLM, one can distinguish two types of chains: the nonlinear Schrödinger (NLS), like lattices, and the nonlinear Klein-Gordon models. In the present paper we concentrate only on the NLS-like models. The statement of the problem as well as some general results are presented in Sec. II. In Sec. III we apply the theory to three particular cases of the deformable discrete nonlinear Schrödinger equation (DNLS) [11], to the model describing Frenkel excitons in a chain with two-level atoms [12], and to the model describing the Heisenberg ferromagnetic [13] within the framework of the stationary phase approximation. In the particular case of the Frenkel exciton lattice, we will obtain DLM in a form of *dark compactons* which are *exact solutions* of the model, which corresponds to deviation of only two central atoms.

II. GENERAL APPROACH AND MODELS

A. Statement of the problem

Let us consider a lattice having the form of a discrete NLS-like equation

$$i\frac{d\mu_n}{dt}+\mu_{n+1}+\mu_{n-1}=V[\{\mu_n,\overline{\mu_n}\}], \tag{1}$$

where in a generic situation $V[\{\mu_n,\overline{\mu_n}\}]$ is a function of $\mu_{n\pm m}$ $(m=0,1,\dots)$,

$$\{\mu_n,\overline{\mu_n}\}=\dots;\mu_{n-1},\overline{\mu_{n-1}};\mu_n,\overline{\mu_n};\mu_{n+1},\overline{\mu_{n+1}};\dots$$

possessing the properties

$$V[\{e^{i\varphi}\mu_n,e^{-i\varphi}\overline{\mu_n}\}]=e^{i\varphi}V[\{\mu_n,\overline{\mu_n}\}] \tag{2}$$

when φ is real, and

$$V[\dots;\mu_{n-m},\overline{\mu_{n-m}};\dots;\mu_{n+m},\overline{\mu_{n+m}};\dots]$$
$$=V[\dots;\mu_{n+m},\overline{\mu_{n+m}};\dots;\mu_{n-m},\overline{\mu_{n-m}};\dots] \tag{3}$$

(a bar stands for the complex conjugation).

The linear part of Eq. (1) is nothing but spatial discretization of the Schrödinger equation and that is why in what follows we refer to $V[\;]$ as a nonlinear potential.

Equation (1) is subject to nonzero boundary conditions

$$\lim_{n\to\pm\infty}\mu_n=\pm\rho\kappa^n e^{i\omega_{\kappa\rho}t}. \tag{4}$$

Here $\kappa=\pm1$: it corresponds either to the center ($\kappa=1$) or to the boundary ($\kappa=-1$) of the BZ, ρ is a positive constant playing the role of the amplitude of the background, and the frequency is given by $\omega_{\kappa\rho}=2\kappa-\omega_0$, where

$$\omega_0=\rho^{-1}\kappa^n V[\{\kappa^n\rho,\kappa^n\rho\}] \tag{5}$$

is a frequency of the underline linear lattice in the ceneter of the BZ (see below). In what follows, finiteness of ω_0 in the limit $\rho\to0$ will be imposed.

B. Stability of the background

Since we are looking for the solutions of Eq. (1) localized against the background, for the first step we have to study the linear stability of the background. To this end we make a substitution

$$\mu_n=\kappa^n e^{i\omega_{\kappa\rho}t}(\rho+\alpha_n), \tag{6}$$

where $\alpha_n=\alpha\exp(i\Omega t-iKn)$ and $\alpha=o(\rho)$ in Eq. (1). Linearizing the so-obtained equation with respect to α_n and using Eq. (3), we get the following dispersion relation $\Omega=\Omega(K)$ associated with the linear evolution of α_n:

$$\Omega^2=\left[-2\kappa(1-\cos K)+\omega_0-a_0-2\sum_{m=1}^{\infty}a_m\cos(Km)\right]^2-\left[b_0+2\sum_{m=1}^{\infty}b_m\cos(Km)\right]^2, \tag{7}$$

where

$$a_m=\left.\frac{\partial V[\{\mu_n(\alpha_n),\overline{\mu_n}(\alpha_n)\}]}{\partial\alpha_{n+m}}\right|_{\alpha_j=0},$$

$$b_m=\left.\frac{\partial V[\{\mu_n(\alpha_n),\overline{\mu_n}(\alpha_n)\}]}{\partial\overline{\alpha}_{n+m}}\right|_{\alpha_j=0}. \tag{8}$$

For the background to be stable, $\Omega(K)$ must be real. As is evident, this happens subject to the conditions as follows:

$$\left[-2\kappa(1-\cos K)+\omega_0-a_0-2\sum_{m=1}^{\infty}a_m\cos(Km)\right]^2>\left[b_0+2\sum_{m=1}^{\infty}b_m\cos(Km)\right]^2, \tag{9}$$

which must be satisfied for all K.

C. Criterion for the existence

The condition of the stability of the background, in the form presented above, does not give yet an explicit condition of the DLM existence. In order to obtain it, we have to specify the name *dark localized mode*. In what follows it is used for a solution of a nonlinear lattice which *exponentially* (*or more rapidly*) tends to the background when $|n|$ goes to infinity. In other words, DLM are determined by the asymptotics

$$\mu_n=\kappa^n(\pm\rho+\psi_n^{(\pm)})e^{i\omega_{\kappa\rho}t},$$
$$\psi_n^{(\pm)}=\psi_\pm e^{-\lambda_\pm|n|}+o(e^{-\lambda_\pm|n|})\quad\text{at}\quad n\to\pm\infty, \tag{10}$$

where $\lambda_\pm$ are real decrements and $\psi_\pm$ are constants (they can be different at $\pm\infty$).

Let us look for a stationary solutions of Eq. (1) in the form

$$\mu_n=\kappa^n e^{i\omega_{\kappa\rho}t}\xi_n, \tag{11}$$

where ξ_n is a real function of the site number only. It solves the equation

$$\omega_0\xi_n-\kappa(2\xi_n-\xi_{n+1}-\xi_{n-1})=\kappa^n f(\{\kappa^n\xi_n\}), \tag{12}$$

where $f(\{\kappa^n\xi^n\})\equiv V[\{\kappa^n\xi_n,\kappa^n\xi_n\}]$. As a matter of fact, Eq. (11) will be the main equation for the next consideration.

Consider now asymptotic $n\to\infty$ (the opposit limit $n\to-\infty$ can be treated similarly) and for the sake of simplicity restrict the consideration to the case of nearest-neighbor interactions (then f depends only on ξ_n and $\xi_{n\pm1}$). In the asymptotic region one can represent $\xi_n=\rho+\psi_n$. Considering the limiting transition $n\to\infty$, it is a direct algebra to ensure that

$$\cosh\lambda_+=\frac{2-\kappa\omega_0+\kappa f_0}{2(1-f_1)}, \tag{13}$$

where $(j=0,1)$

$$f_j=\left.\frac{\partial f(\{x_n\})}{\partial x_{n+j}}\right|_{x_n=\kappa^n\rho,x_{n\pm1}=\kappa^{n+1}\rho}.$$

Equation (13) determines the decrement of the DLM decay. As is evident, for the existence of the DLM the following condition must be satisfied:

$$\frac{2-\kappa\omega_0+\kappa f_0}{2(1-f_1)}>1. \tag{14}$$

Requirement (14) together with Eq. (9) make up the necessary conditions for the DLM existence (in the case of nearest-neighbor interactions). These formulas, however, are not independent. They both can be obtained directly from Eq. (7). Moreover, for the class of the potentials considered in the present paper, requirement (14) as well as its generalization to the case of long-range interactions are always satisfied provided that Eq. (9) is satisfied.

Indeed, let us introduce the notation $z=\cos K$ and assume that the upper limit in the sums in Eq. (7) is M (this means that M neighbors of each atom interact with it). Then Eq. (7) can be rewritten in the form

$$\Omega^2-\Lambda\prod_{j=1}^{2M}(z-z_j)=0, \tag{15}$$

where Λ and z_j $(j=1,\ldots,2M)$ are expressed through the parameters of the problems (i.e., through the coefficients a_m and b_m). The existence of a DLM means that (i) the background is stable and (ii) there exist ''stationary'' (i.e., corresponding $\Omega=0$) linear excitations which are spatially unstable. The first requirement means that the polynomial

$$\Lambda\prod_{j=1}^{2M}(z-z_j)$$

is positively defined for all $z\in[-1,1]$. In that case, no roots z_j can be placed inside the interval $[-1,1]$. Thus $|z_j|\geq1$. On the other hand, the existence of at least one root z_j which modulus is larger than 1 means the existence of a spatially unstable stationary excitation.

III. LATTICE PATTERN

One of the main consequences of the above consideration is that the frequency $\omega_{\kappa\rho}$ and hence the width (it is related to $\lambda_\pm$) of the dark localized modes are not arbitrary but uniquely determined by the nonlinear potential and by the amplitude of the background. This means, in particular, that in a generic case it is impossible to reach strong localization similar to that in the conventional ''bright'' case [5] for any ρ. This causes a problem of determination of the lattice pattern, which generally speaking can be solved only by numerical methods. In the meantime, one can pose a question about the region of parameters allowing strong localization and about the number of sites to be taken into account, in order to get the necessary accuracy for a given ρ. In this section we consider these questions as well as the explicit lattice pattern of DLM.

To this end we define a ''shifted'' atom as an atom which displacement is neither zero nor ρ and consider the cases of DLM centered on an atom (case 1) and centered between two atoms (case 2) separately. Schematically the lattice patterns corresponding to these cases are depicted in Figs. 1(a)–1(d), respectively. In case 1 we use the terminology as follows: the n_0th approximation means that atoms $\pm1,\pm2,\ldots,\pm n_0$ are shifted while $|\xi_n|=\rho$ at $|n|>n_0$. In case 2, the n_0th approximation means that atoms at sites $-n_0+1,\ldots,0,1,\ldots,n_0$ are shifted while $|\xi_n|=\rho$ at $n>n_0$ and $n\leq-n_0$.

Next, we observe that Eq. (12) subject to the boundary conditions $\xi_n\to\pm\rho$ as $n\to\pm\infty$ possesses the ''integral''

$$I\equiv\omega(\xi_n\xi_{n+1}-\rho^2)+\kappa(\xi_{n+1}-\xi_n)^2+\sum_{k=n+1}^{\infty}\kappa^k f(\{\kappa^k\xi_k\})(\xi_{k+1}-\xi_{k-1})=0. \tag{16}$$

The usefulness of the preceding formula is justified by the fact that within the framework of the n_0th approximation each term in Eq. (16) equals zero identically for $|n|>|n_0|$.

In order to discuss another important consequence of Eq. (16), we introduce the name *dark compacton* (by analogy with the conventional compactons) for the solutions in which all atoms with $|n|\geq n_0+1$ in case 1 and $n>n_0$ and $n\leq1-n_0$ in case 2 have amplitudes equal to ρ, i.e., displacements equal to zero. Then it follows from Eq. (16) that there exist no dark compactons in the case when the nonlinear potential is a linear function of $\mu_{n\pm1}$. Indeed, let us consider case 1 and take into account that for the solution mentioned above (if any) the n_0th approximation must lead to the exact result. Then it follows from Eq. (16) that there must be satisfied the relation

$$f(\{\kappa^{n_0+1}\rho\})-f(\{\kappa^{n_0+1}\xi_{n_0+1}\})=\kappa^{n_0}(\rho-\xi_{n_0}). \tag{17}$$

Subject to suppositions that ξ_n is a dark compacton, Eq. (17) is an equation with respect to the only variable ξ_{n_0}. If $V[\{\mu_n,\overline{\mu_n}\}]$ is a linear function with respect to $\mu_{n\pm1}$ (notice that the potential is allowed to be nonlinear with respect to μ_n), then Eq. (17) is a linear equation with respect to ξ_{n_0} and its unique solution is $\xi_{n_0}=\rho$.

As will be shown below, dark compacton solutions are possible for potentials $V[\{\mu_n,\overline{\mu_{n\pm1}}\}]$, which are nonlinear with respect to $\mu_{n\pm1}$.

We use integral (16) as a criterion for the validity of the n_0th approximation, and, hence, as a criterion for the localization [a solution of Eq. (12) solves also Eq. (16)]. Thus we compute a solution of Eq. (1) in the n_0th approximation and then evaluate the integral I corresponding to such a solution.

Although the approach has a generic character, for the sake of definiteness below the results are applied to the models as follows.

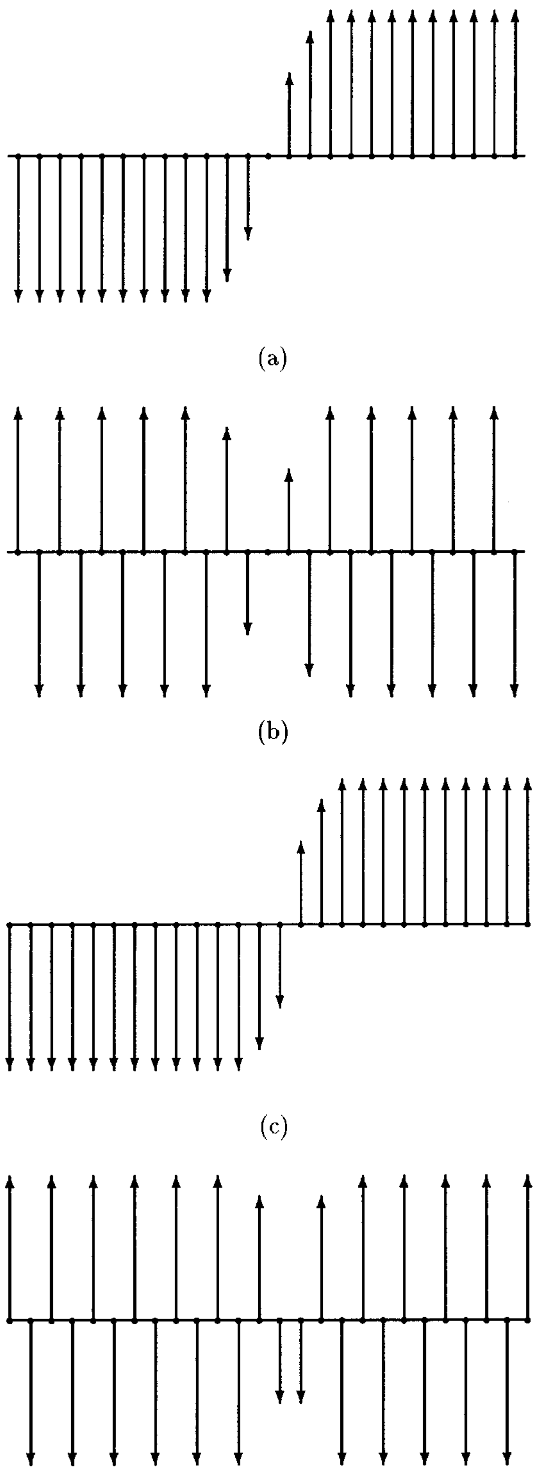

FIG. 1. Schematic representation of a lattice pattern in the cases of dark localized mode centered on a site at (a) $\kappa=1$, (b) $\kappa=-1$ and centered between sites (c) $\kappa=1$, (b) $\kappa=-1$.

(i) The deformable discrete NLS equation (DNLS) [11],

$$V[\{\mu_n,\overline{\mu_n}\}]=V_{\rm NLS}=\kappa_{nl}\epsilon|\mu_n|^2(\mu_{n+1}+\mu_{n-1})+2\kappa_{nl}(1-\epsilon)|\mu_n|^2\mu_n, \tag{18}$$

which reduces to the integrable AL model [3,9] at $\epsilon=1$ and to the so-called self-trapping model at $\epsilon=0$ ($\kappa_{nl}=\pm1$ characterizes the type of the nonlinearity).

(ii) A model describing Frenkel excitons in a chain of two-level atoms with exchange interaction in the SU(2) coherent state representation [12],

$$V[\{\mu_n,\overline{\mu_n}\}]=V_F=\frac{(\mu_{n+1}^2+\mu_n^2)\overline{\mu}_{n+1}}{1+|\mu_{n+1}|^2}+\frac{(\mu_{n-1}^2+\mu_n^2)\overline{\mu}_{n-1}}{1+|\mu_{n-1}|^2}+\eta\mu_n\left(\frac{|\mu_{n+1}|^2}{1+|\mu_{n+1}|^2}+\frac{|\mu_{n-1}|^2}{1+|\mu_{n-1}|^2}\right). \tag{19}$$

Here the parameter η describes the relation between exciton-exciton and exchange interactions ($\eta>0$).

(iii) A model describing the Heisenberg ferromagnetic in the stationary phase representation [13],

$$V[\{\mu_n,\overline{\mu_n}\}]=V_H=\frac{(\mu_{n+1}^2+\mu_n^2)\overline{\mu}_{n+1}}{1+|\mu_{n+1}|^2}+\frac{(\mu_{n-1}^2-\mu_n^2)\overline{\mu}_{n-1}}{1+|\mu_{n-1}|^2}+\eta\mu_n\left(\frac{1-|\mu_{n+1}|^2}{1+|\mu_{n+1}|^2}+\frac{1-|\mu_{n-1}|^2}{1+|\mu_{n-1}|^2}\right). \tag{20}$$

Here η is the anisotropy of the exchange.

As has been mentioned, the conventional nonlinear intrinsic modes are localized on a very few atoms [4] and this fact allows one to employ the Green-function method in order to estimate displacements of atoms analytically [5]. In case 1 of DLM this corresponds to the first approximation, when only three central atoms are placed out of the background and atoms with $|n|\geqslant2$ are characterized by the amplitude ρ. The same approximation in case 2 correspond to only two shifted atoms with $n=0$ and $n=1$. In what follows we concentrate on the study of the validity of theses approximations.

A. DNLS equation

We start with the DNLS model. First, we recall that $\epsilon=1/\rho^2$ is a "singular" point of the model [14]. That is why in what follows the consideration is restricted to the parameters satisfying the condition

$$\epsilon\rho^2<1. \tag{21}$$

Consider the case 1 at $\kappa_{\rm nl}=1$. By direct calculus one obtains that the background stability condition (9) [and, hence, Eq. (14)] is satisfied for any ϵ in the center of BZ ($\kappa=1$) and for $\epsilon>\frac{1}{2}$ if $\kappa=-1$. In the case $\kappa_{\rm nl}=-1$, DLM can exist only at the boundary of BZ and at $\epsilon<\frac{1}{2}$.

The results of the numerical study of the integral (16) are represented in Fig. 2 for case 1 in the center (a) and boundary (b) of BZ ($\kappa=1$ and $\kappa=-1$, respectively). One can observe that the curves go to zero as ρ increases. This is related to decreasing the region of localization with growth of the background amplitude (for the integrable AL model this result can be obtained analytically [9]). Thus only in a narrow region near $\epsilon\rho^2=1$ can one consider DLM as highly localized (i.e., well described in the first approximation). The

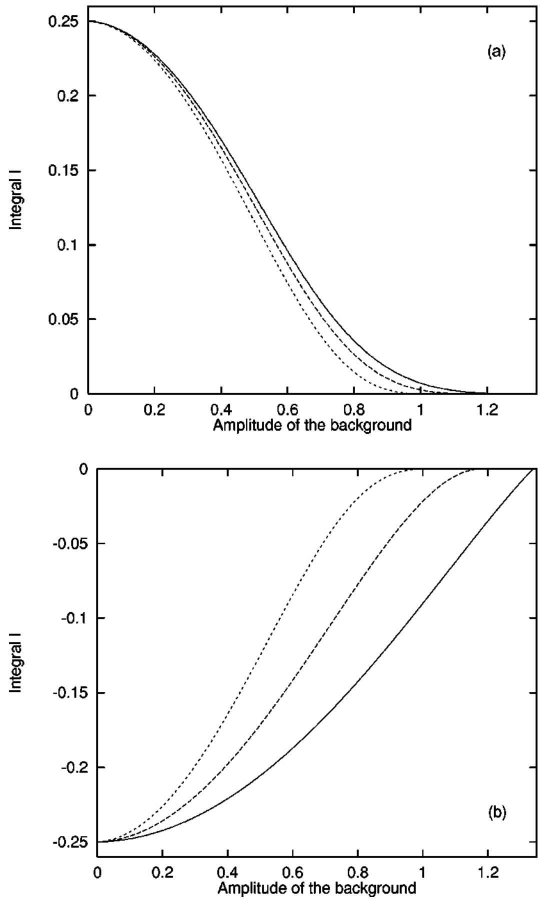

FIG. 2. Integral I of DNLS equation vs the background amplitude (in dimensionless units) for the center (a) and the boundary (b) of BZ [it corresponds to the patterns of Figs. 1(a) and 1(b), respectively]. Solid, broken, and dashed lines correspond to $\epsilon=0.55$, $\epsilon=0.7$, and $\epsilon=0.93$. DLM is centered on a site. In all the figures, $\kappa_{\rm nl}=1$.

small background corresponds to the continuum limit. Then I receives a large value equal for all ϵ (the continuum limit of the DNLS equation does not depend on ϵ).

In Figs. 3(a)–3(c) we present positions of the first and second shifted atoms computed within the framework of the first and second approximations as functions of the background amplitude. When $\rho\rightarrow1$ in Figs. 3(a) and 3(b) one observes good convergence of the approximations and a tendency of the position of the second atom to 1. This reflects the stronger localization at higher amplitudes of the background. In the meantime, the character of the growth of the amplitude with the background is different in the center [Fig. 3(a)] and on the boundary [Fig. 3(b)] of the first BZ. This is explained by stronger localization in the center of the BZ [see Fig. 3(a)] compared with the one at the boundary. Delocalization of the pulse, i.e., failure of the first approximation for all ρ, occurs in the vicinity of the critical point $\epsilon=\frac{1}{2}$ [see Fig. 3(c)]. All the figures display the same feature: *the higher the localization, the larger the modulus of the displacement of the atoms*. It is to be mentioned here that for obtaining the lattice pattern we always choose roots of the respective equations, placed between zero and the background amplitude.

Displacements of the atoms corresponding to DLM centered between two sites (case 2) are represented in Fig. 4. We

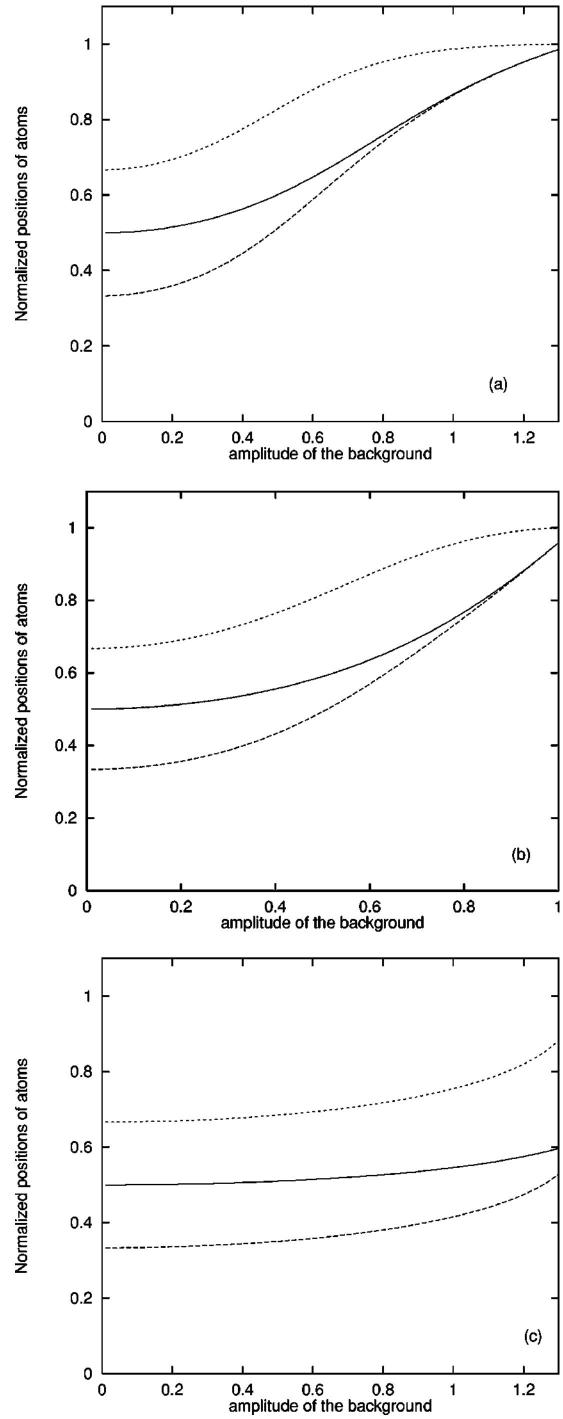

FIG. 3. Positions of the first atom computed within the framework of the first approximation (solid line) and second approximation (broken line) for DLM of the DNLS model centered on a site. The dashed line displays the position of the second shifted atom in the second approximation. All positions are normalized to the amplitude of the background. (a)–(c) correspond to $\epsilon=0.55$, $\kappa=1$; $\epsilon=0.93$, $\kappa=-1$; and $\epsilon=0.55$, $\kappa=-1$. In all the figures, $\kappa_{\rm nl}=1$.

observe qualitatively different behavior of the first atom displacements versus background amplitude and slower convergence compared with case 1. The amplitude of the oscillations of a first atom is smaller when DLM are centered between sites compared with one in the case of DLM centered on a site. As in case 1, in the region near $\epsilon=\frac{1}{2}$ the first approximation fails for all ρ [Fig. 4(d)].

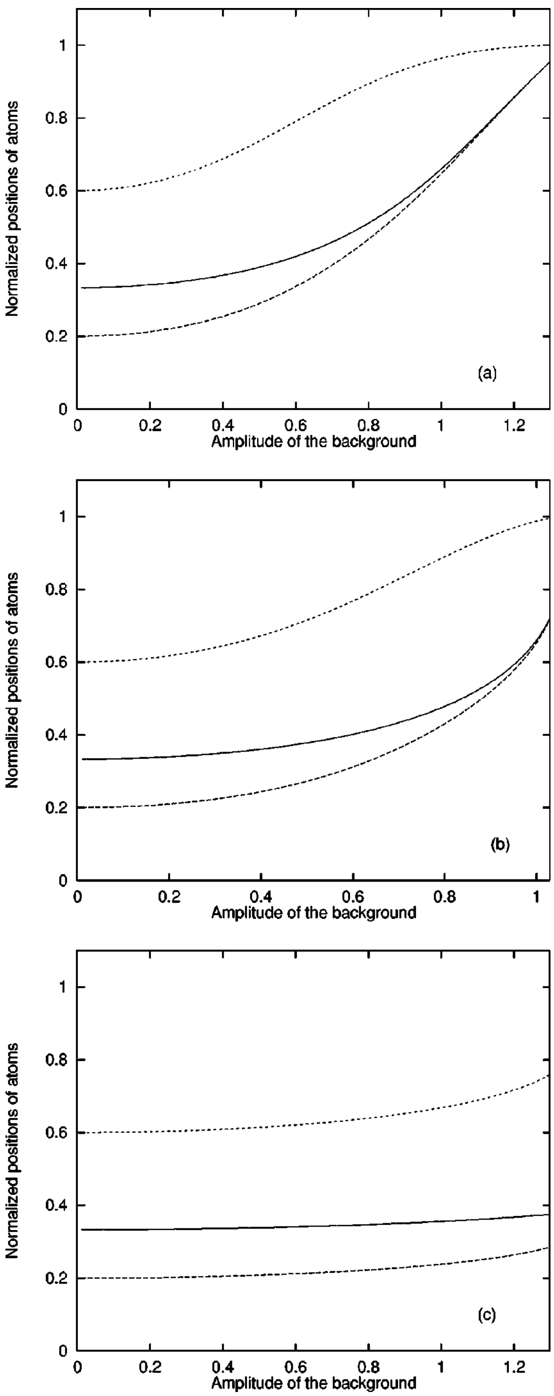

FIG. 4. Position of the first atom computed within the framework of the first approximation (solid line) and second approximation (broken line) for DLM of the DNLS model centered between two sites. The dashed lines display the position of the second shifted atom in the second approximation. All positions are normalized to the amplitude of the background. (a)–(c), correspond to $\epsilon = 0.55$, $\kappa = 1$; $\epsilon = 0.55$, $\kappa = -1$; and $\epsilon = 0.9$, $\kappa = -1$. In all the figures, $\kappa_{\rm nl} = 1$.

B. Dark compactons of the Frenkel exciton model

Let us consider now the model (20) which describes Frenkel excitons in a chain of two-level atoms. Stability of the background essentially depends on the value η (it is established in [12]). Here we restrict the analysis only to the case $\eta < 2$, which corresponds to the stable background at $\kappa = 1$

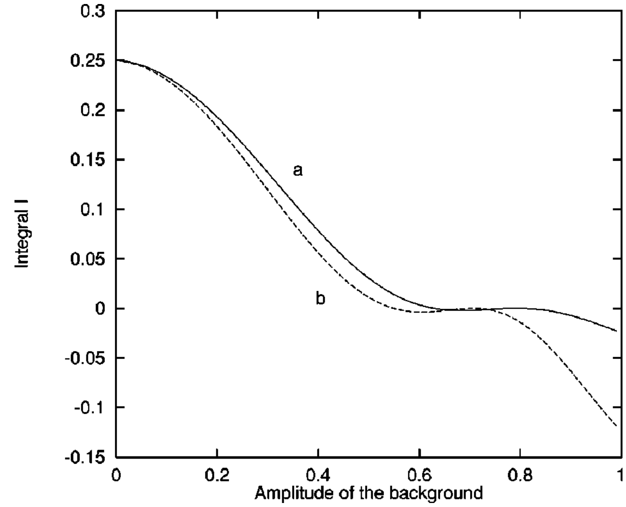

FIG. 5. Integral I for the Frenkel exciton model vs the background amplitude (in dimensionless units) in the center of BZ for (a) $\eta = 0.55$, (b) $\eta = 0.95$. The results are given for the case of DLM centered on a site.

and $\rho < 1$, which is most interesting from the physical point of view. In Fig. 5 we present two curves corresponding to integral (16) for different values of η. A peculiarity of the figure is that at some values ($\rho \approx 0{,}635$ for $\eta = 0.55$ and $\rho \approx 0.555$ for $\eta = 0.95$) the integral I becomes zero. This means that the first approximation gives an exact result. In order to understand the behavior of the solutions in the mentioned cases, in Fig. 6 we present shifts of the first and second atoms corresponding to $\eta = 0.95$. From the figure one can see that at $\rho \approx 0.555$ the two approximations give exactly the same results (characterized by the amplitude ρ of the displacement of the second atom) and at bigger amplitudes of the background the displacement of the second atom becomes larger than the background amplitude.

Thus there must exist *exact* solutions of the Frenkel exciton model such that only two atoms (with $n = \pm 1$) are

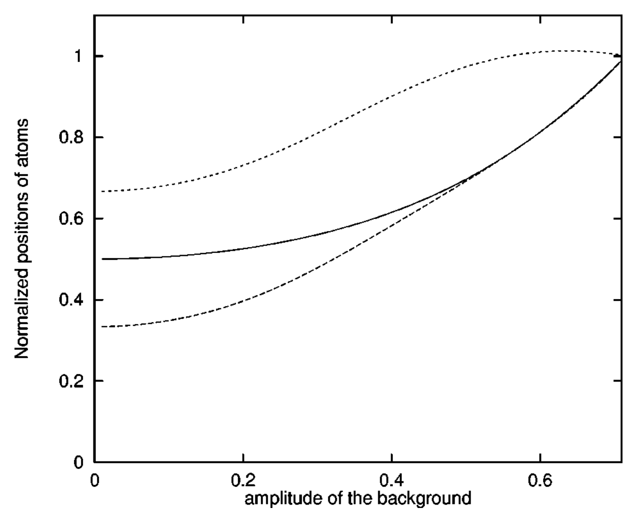

FIG. 6. Position of the first atom computed within the framework of the first approximation (solid line) and second approximation (broken line) for DLM of the Frenkel exciton model at $\eta = 0.95$ centered on a site. The dashed lines display the position of the second shifted atom in the second approximation. All positions are normalized to the amplitude of the background.

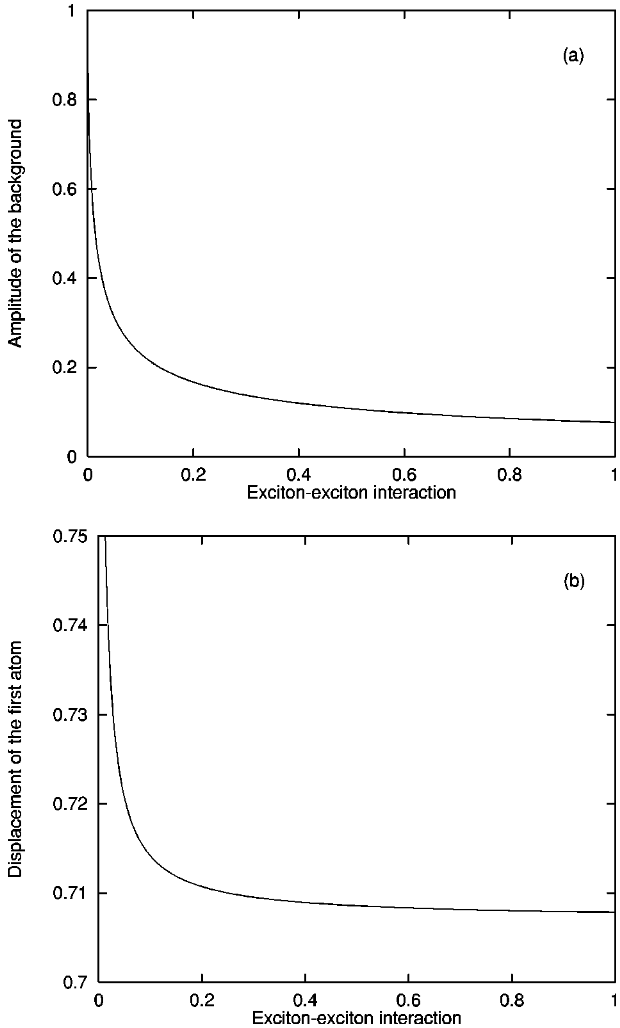

FIG. 7. Dependence ρ vs η resulting in dark compactons centered on a site (a). Amplitude of a dark compacton vs exciton-exciton interaction (b) in dimensionless units.

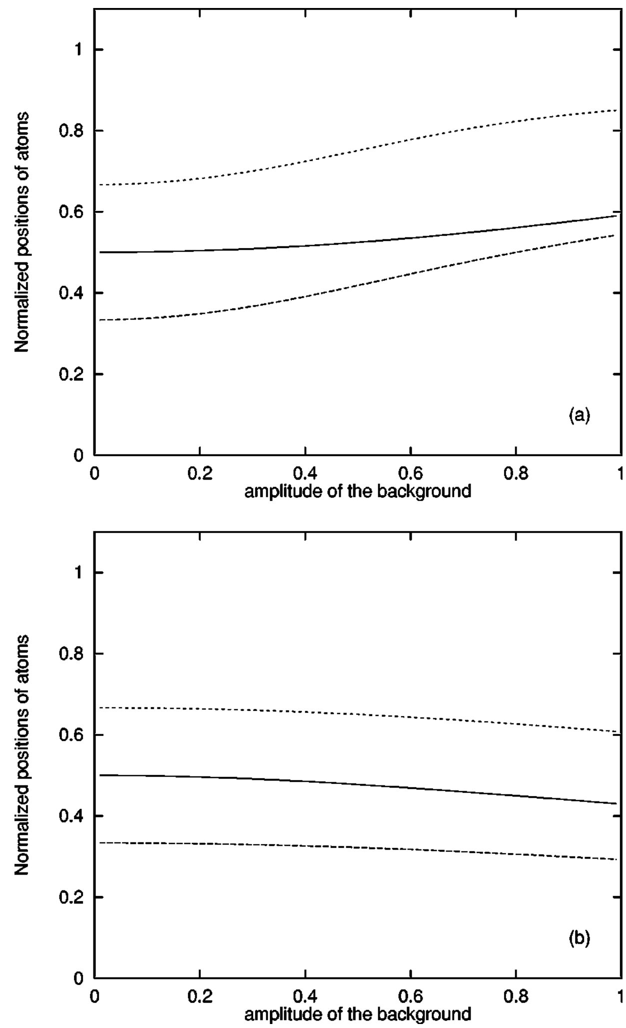

FIG. 8. Position of the first atom computed within the framework of the first approximation (solid line) and second approximation (broken line) for DLM of the Heisenberg ferromagnetic model centered on a site. The dashed lines display the position of the second shifted atom in the second approximation. All positions are normalized to the amplitude of the background. Plots (a) and (b) correspond to $\eta=0.55$ and $\eta=0.95$.

shifted. According to the definition given above such solutions can be identified as *dark compactons* [15] (centered on a site a solution with compact support)[16]. The compacton has the amplitude of the background fixed by the exciton-exciton interaction η. Respective dependence is computed as the root of the cubic polynomial (which is obtained from the dynamical equations for $n=1$ and $n=2$). In Fig. 7(a) we represent the dependence ρ versus η. Then the amplitude of the displacement of the first atom is given by the formula

$$\xi_1=\frac{\rho(1-\rho^2)(2+\eta)}{\rho^4(1+\eta)-\rho^2(4+3\eta+\eta^2)+3+2\eta}. \quad (22)$$

The dependence of the displacement ξ_1 on the exciton-exciton interaction η is depicted in Fig. 7(b).

C. Heisenberg ferromagnetic model

At $\eta<1$ and arbitrary ρ the ferromagnetic model (20) has a stable background only in the center of BZ: hence in what follows we deal only with the case $\kappa=1$. It is to be pointed out that formally Eq. (20) does not result in any mathematical restriction on the amplitude of the background. In the meantime, from the physical point of view the situation when the system is rather close to the vacuum state (i.e., to the state $\mu_n\equiv 0$) is of the main interest. Respectively, below we restrict the consideration to the range $0<\rho<1$.

Figure 8 displays the positions of the atoms in the first and second approximations. In the figures one can see that for almost all $\eta<1$ the first approximation fails. Moreover, as the anisotropy constant grows, the second approximation fails as well (the respective integral I becomes a growing function). In Fig. 8(b) this is reflected in the fact that all the functions are decaying.

IV. CONCLUSION

To conclude, various lattice patterns corresponding to DLM (or kinks) of discrete NLS-like models have been considered. Only in a rather narrow region at large amplitudes of the background does one observe strong localization of modes, allowing one to treat them within the framework of the approximation of only two shifted atoms. This is related to the fact that the small amplitude limit corresponds to the continuum limit. NLS-like models studied above possess at least one integral of motion deviation, which from zero can be used as a criterion for delocalization.

Three lattices have been considered as examples. It has been found that they display essentially different solutions which depend on the type of the nonlinear potential. The DNLS model possesses DLM solutions at large enough amplitudes of the background. The Frenkel exciton model has a new type of exact solutions, dark compactons. The Heisenberg ferromagnetic model does not have strongly localized dark modes (at least in the physically relevant region of the parameters).

The modes studied above are static. Their dynamical properties we leave for further investigation. In this context we notice that compared with the case of intrinsic localized modes which could be movable at small enough amplitudes (see, e.g., [6]), in the case at hand one has fewer free parameters and thus one should not expect the possibility of directed motion of highly localized DLM.

ACKNOWLEDGMENTS

The work of V.V.K. has been supported by FEDER, and by the Program PRAXIS XXI under Grant No. PRAXIS/2/2.1/FIS/176/94.

[1] A. Tsurui, Prog. Theor. Phys. **48**, 1196 (1972); M. Remoissenet, Phys. Rev. B **33**, 2386 (1986); K. Yoshimura and S. Watanabe, J. Phys. Soc. Jpn. **60**, 82 (1991); V. V. Konotop, Phys. Rev. E **53**, 2843 (1996).

[2] H. Flashka, Phys. Rev. B **9**, 1924 (1974); S. V. Manakov, Zh. Éksp. Teor. Fiz. **67**, 543 (1974) [Sov. Phys. JETP **40**, 269 (1974)].

[3] M. Ablowitz and J. Ladik, J. Math. Phys. **16**, 598 (1975); **17**, 1011 (1976).

[4] A. J. Sievers and S. Takeno, Phys. Rev. Lett. **61**, 970 (1988); S. Takeno, K. Kisoda, and A. J. Sievers, Prog. Theor. Phys. Suppl. **94**, 242 (1988).

[5] S. Takeno, J. Phys. Soc. Jpn. **61**, 2821 (1992).

[6] V. V. Konotop and S. Takeno, Phys. Rev. E **54**, 2010 (1996).

[7] R. S. MacKay and S. Aubry, Nonlinearity **7**, 1623 (1994); D. Bambusi, *ibid.* **9**, 433 (1996).

[8] K. Yoshimura and S. Watanabe, J. Phys. Soc. Jpn. **60**, 82 (1991).

[9] V. V. Konotop and V. E. Vekslerchik, J. Phys. A **25**, 4037 (1992); V. E. Vekslerchik and V. V. Konotop, Inverse Probl. **8**, 889 (1992).

[10] B. Denardo, B. Galvin, A. Greenfield, A. Larraza, S. Putterman, and W. Wright, Phys. Rev. Lett. **68**, 1730 (1992).

[11] M. Salerno, Phys. Rev. A **46**, 6856 (1992); Phys. Lett. A **162**, 381 (1992).

[12] V. V. Konotop and S. Takeno, Phys. Rev. B **55**, 11 342 (1997).

[13] V. V. Konotop, M. Salerno, and S. Takeno, Phys. Rev. B **58**, 14 892 (1988).

[14] V. V. Konotop and M. Salerno, Phys. Rev. E **55**, 4706 (1997).

[15] We notice that compacton solutions do not exist in nonlinear Klein-Gordon lattices due to the evident reason that their existence would contradict the Newton laws.

[16] Ph. Rosenau, and J. M. Myman, Phys. Rev. Lett. **70**, 564 (1993).